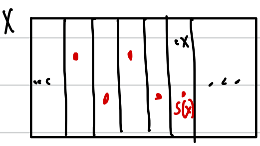
Measure Theory with Ergodic Horizons
Lecture 7

Non-measurable sets.

We will give an example of a non-measurable subset of $\mathbb{R}$ with respect to the Lebesgue measure λ, and outline in HW how to build a non-meas. set for the Bernoulli measures.

Def. For an equiv. rel. E on a set X, a *selector* for E is a function $s : X \to X$ such that $s(x) \in [x]_E$ and $x E y \iff s(x) = s(y)$. A *transversal* for E is a set $S \subseteq X$ that intersects every E-class in exactly one point.



Note. Having a selector s, one sees that $s(X)$ is a transversal, and vice versa, if S is a transversal, then $s : X \to X$ is a selector, defined by $x \mapsto$ the unique element $y \in [x]_E \cap S$.
A selector (and hence a transversal) always exists by axiom of choice.
However, this usually results in ill-behaved sets and functions.

Example.
Let E_v be the so-called *Vitali equivalence relation* on $\mathbb{R}$ defined by

$$x \, E_v \, y \;:\iff\; x - y \in \mathbb{Q}.$$

In other words, this is the coset equivalence relation of $\mathbb{Q}$ as a subgroup of $\mathbb{R}$, also the orbit equiv. rel. of the translation action $\mathbb{Q} \curvearrowright \mathbb{R}$.
Let S be a transversal for $E_v|_{[0,1]}$, i.e. $S \subseteq [0,1]$ that meets every E_v-class at exactly one point. We show that S is not measurable wrt the Lebesgue meas λ.
Suppose it is measurable. Then note that

$$[0,1] \subseteq \bigsqcup_{q \in \mathbb{Q} \cap [-1,1]} q + S \subseteq [-1,2]$$

But then $1 = \lambda([0,1]) \le \lambda\Big(\bigsqcup_{q\in\mathbb{Q}\cap[-1,1]} q+S\Big) = \sum_{q\in\mathbb{Q}\cap[-1,1]} \lambda(q+S) = \sum_{q\in\mathbb{Q}\cap[-1,1]} \lambda(S) = \infty\cdot\lambda(S) \le \lambda([-1,2]) = 3$

So $\lambda(S)$ has to be 0 because $\infty\cdot\lambda(S) \le 3$ but it can't be 0 because $1 \le \infty\cdot\lambda(S)$. $\square$

Remark. It is tempting to think that only axiom of choice can give non-measurable sets but this is not quite true. By definition, we know that all Borel subsets of $\mathbb{R}^d$ are λ-measurable. It is one of the first theorems of descriptive set theory, that projections of Borel sets are still λ-measurable; these sets are called *analytic*. Hence their complements, called *coanalytic*, are also measurable. What about projections of coanalytic sets, are they measurable? It turns out that the answer to this question is independent from ZFC. More precisely, there is a particular G_δ subset $B \subseteq \mathbb{R}^3$, such that the measurability of the set $\mathrm{proj}_{\mathbb{R}}(\mathrm{proj}_{\mathbb{R}^2} B)^c$ is independent from ZFC.

Pocket tools for working with measures.

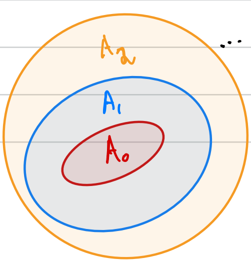
Prop (monotone convergence). Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) $\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to\infty} \mu(A_n)$ for all μ-measurable A_n with $A_n \subseteq A_{n+1}$.

(b) If $\mu(X) < \infty$ then $\mu\Big(\bigcap_{n\in\mathbb{N}} B_n\Big) = \lim_{n\to\infty} \mu(B_n)$.

Caution. Part (b) can fail for infinite measures: let $B_n := [n, \infty)$ so $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$ but $\lim_{n\to\infty} \lambda(B_n) = \lim_{n\to\infty} \infty \ne 0 = \lambda(\emptyset)$.

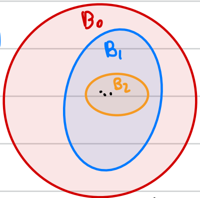
Proof. (a)



... We disjointify: $\tilde{A}_0 := A_0$, $\tilde{A}_n := A_n \setminus A_{n-1}$. Then $\bigsqcup_n \tilde{A}_n = \bigcup_n A_n$ and

$$\mu\Big(\bigcup_n A_n\Big) = \mu\Big(\bigsqcup_n \tilde{A}_n\Big) = \sum_n \mu(A_n \setminus A_{n-1}) = \lim_{n\to\infty} \sum_{k\le n} \mu(A_k \setminus A_{k-1})$$

$$\lim_{n\to\infty} \mu\Big(\bigsqcup_{k\le n} A_k \setminus A_{k-1}\Big) = \lim_{n\to\infty} \mu(A_n).$$ $\boxed{a}$

(b) 

Let $A_n := B_0 \setminus B_n$, then (A_n) is increasing, so by (a), we have $\mu(\bigcup_n A_n) = \lim_{n\to\infty} \mu(A_n)$. (*)

But $\bigcup_n A_n = B_0 \setminus \bigcap_n B_n$ and $\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) = \mu(B_0)$, hence $\mu(A_n) = \mu(B_0) - \mu(B_n)$ because $\mu(B_n) < \infty$. Also, $\mu(\bigcup_n A_n) = \mu(B_0 \setminus \bigcap_n B_n) = \mu(B_0) - \mu(\bigcap_n B_n)$ again because $\mu(\bigcap_n B_n) < \infty$. Thus,

by (*), $\mu(B_0) - \mu(\bigcap_n B_n) = \lim_{n\to\infty}\big(\mu(B_0) - \mu(B_n)\big)$, so $\mu(\bigcap_n B_n) = \lim_{n\to\infty} \mu(B_n)$. $\boxed{b}$

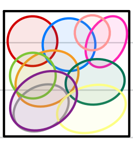
Borel-Cantelli lemmas (Important Pigeonhole Principles). Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) Let (A_n) be a sequence of μ-measurable sets with summable measures, i.e. $\sum_n \mu(A_n) < \infty$. Then the set

$$\limsup_{n\to\infty} A_n := \Big\{ x \in X : \overset{\forall m\, \exists n \ge m}{\exists^\infty_n}\; x \in A_n \Big\}$$

is μ-null. In other words, the unlucky points form a null set.

(b) (Measure comparison) Suppose $\mu(X) < \infty$. Let (A_n) be a sequence of meas. sets such that $\exists\, \delta > 0$ with $\mu(A_n) \ge \delta$ for all $n \in \mathbb{N}$.

X 

Then $\limsup_{n\to\infty} A_n$ has measure $\ge \delta$.

Proof. Note that $\limsup_{n\to\infty} A_n = \bigcap_m \bigcup_{n \ge m} A_n$, so $\mu(\limsup_n A_n) \overset{\forall m}{\le} \mu\Big(\bigcup_{n\ge m} A_n\Big) \le \sum_{n \ge m} \mu(A_n)$

$\to 0$ as $m \to \infty$ in part (a), so $\mu(\limsup_n A_n) = 0$.

In part (b), $\mu(\limsup_n A_n) = \lim_{m\to\infty} \mu(\bigcup_{n\geq m} A_n)$ by the decreasing monotone convergence,

but $\forall m$ $\mu(\bigcup_{n\geq m} A_n) \geq \mu(A_m) \geq \delta$, so $\lim_{m\to\infty} \mu(\bigcup_{n\geq m} A_n) \geq \delta$. $\square$

Application. Let $(X, \mathcal{B}, \mu)$ be a measure space. A sequence (V_n) of sets is called vanishing (resp. almost vanishing) if it is decreasing, $V_n \supseteq V_{n+1}$, and $\bigcap_{n\in\mathbb{N}} V_n = \emptyset$ (resp. μ-null).

Prop. Let $\mathcal{F}$ be a family of μ-meas. sets that is closed under ctbl unions and contains sets of arbitrarily small measure. Then $\mathcal{F}$ contains an almost vanishing sequence.

Proof. Let $A_n \in \mathcal{F}$ be a set of measure $\leq 2^{-n}$. Let $B_m := \bigcup_{n\geq m} A_n$. (B_m) is decreasing and is in $\mathcal{F}$, moreover it's almost vanishing because $\bigcap_m B_m = \limsup_n A_n$, which is null by Borel-Cantelli. $\square$